Hall Effect, or Hyperbolic Magnetohydrodynamics, HMHD

E. A. Witalis

National Defence Research Institute, S-10254 Stockholm, Sweden

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The MHD theory of present magnetic fusion research is briefly reviewed with emphasis on its mathematically diffusive character. The importance of retaining the Hall effect term, neglected in ideal or resistive MHD theory, is stressed. Elliptic MHD theory is critically dismissed. The *H*all effect, or *H*yperbolic, *M*agneto*H*ydro*D*ynamics, HMHD, is shown to follow as the consequence of a revision of plasma electrodynamics so as to account for the fundamental plasma quasineutrality. The non-validity of Newton's third law in charged particle contexts is then central. Previously poorly understood phenomena, such as plasma edge effects and magnetic field line reconnection are found to be inherent properties in this HMHD plasma description. The "magnetic bottle" principle for high density plasma confinement is shown to be physically unsound because there will exist a no-confinement plasma boundary region with HMHD theory properties. Arguments for non-thermal fusion, provided by HMHD theory, are given.

1. Introduction

According to the usual, single-fluid magnetohydrodynamics (MHD) plasma description a high density and high temperature plasma entity can be confined by a magnetic field. This is, of course, the basic idea of magnetic confinement thermonuclear fusion research. Specifically, a stable equilibrium is thought to exist by the plasma pressure gradient being balanced everywhere by the magnetic Lorentz force

$$\operatorname{grad} p = \boldsymbol{j} \times \boldsymbol{B} . \tag{1}$$

p is the plasma pressure, $\boldsymbol{j}$ the plasma current density and $\boldsymbol{B}$ the magnetic field strength.

An indeterminate or rather negative outcome of this "magnetic bottle" conjecture, extensively tested but in secrecy for nearly one decade, became apparent and public at the Geneva Conference in 1958. A number of magnetic configurations presented there have later been tested further with limited success. Presently, efforts are dominated by the toroidal "Tokamak" bottle, in spite of the fact that an equilibrium according to (1) is mathematically nonexistent for such toroidal configurations [1].

2. The Ion Collisionless Skin Depth

A general experimental observation is the relation between the characteristic scaling length L of the more or less confined plasma and its density n_i [2],

$$\lambda_i^2/L^2 \simeq 1, \qquad \lambda_i^2 = m_i/\mu_0 e^2 n_i . \tag{2}$$

m_i is the ion mass, μ_0 the permeability of free space, e the ion charge. λ_i is usually called the ion collisionless skin depth or ion plasma wave-length but sometimes also denoted the ion inertial length. It expresses an ionic kind of the London penetration depth of superconductor theory, and for plasmas it gives the shielding distance for magnetic effects by ionic currents.

λ_i is not only the generally observed and large field-plasma dynamic interaction distance [3]. In addition, dense plasma entities spontaneously evolve, by particle emissions, filamentation, formation of magnetized whirls etc. so as to attain the condition expressed by (2), [4]. Such phenomena have been studied theoretically very much but almost always in the form deviations ("instabilities", "anomalities", "singularities") from an initial configuration obeying MHD theory. The outcome of these extensive theoretical efforts was recently judged [5] by the need for "a still deeper understanding of the physical phenomena in fusion plasmas. The most obvious of this kind is that of transport in Tokamak plasmas, which has so far resisted all attempts at a generally accepted explanation".

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Reprint requests to Dr. E. A. Witalis, National Defence Research Institute, P.O. Box 27322, S-10254 Stockholm, Sweden.

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3. The Character of MHD Plasma Theory

To derive the physical and mathematical key feature of classical MHD theory, i.e. the *diffusion* of vector fields, one may combine the very simplest version of Ohm's law

$$\boldsymbol{j} = \sigma \boldsymbol{E}, \qquad \sigma = \text{constant} \tag{3}$$

with two generally valid Maxwell equations, in usual SI notations:

$$\operatorname{curl} \boldsymbol{E} = -\partial \boldsymbol{B}/\partial t\,, \tag{4}$$

$$\operatorname{div} \boldsymbol{B} = 0\,, \tag{5}$$

and a third Maxwell equation in its MHD plasma, i.e. low frequency, approximation:

$$\operatorname{curl} \boldsymbol{B} = \mu_0 \boldsymbol{j}\,. \tag{6}$$

The rotational part of (3) then yields

$$\nabla^2 \boldsymbol{B} - \mu_0 \sigma \partial \boldsymbol{B}/\partial t = 0\,, \tag{7}$$

which just describes source-free *diffusion* of the magnetic field strength in a medium with the diffusion constant $(\mu_0\sigma)^{-1}$. With increasing conductivity the diffusion speed is slowed down, to become zero in the limit $\sigma \to \infty$.

An extension of (3) to moving media, mass velocity $\boldsymbol{V}$,

$$\boldsymbol{j} = \sigma(\boldsymbol{E} + \boldsymbol{V} \times \boldsymbol{B}) \tag{8}$$

leads in combination with (4) – (6) to a similar diffusion, decreasing upon the increase of σ, but with reference to the moving mass frame. A limit often referred to is the Alfvén concept of perfect conductivity with the magnetic field lines "frozen" to the moving mass for $\sigma \to \infty$.

The outlined MHD theory provide a well-established description of field dynamics in liquid conductors and metals. In contrast, even for highly conducting plasmas its applicability ranges between acceptable and misleading. This should hardly be surprising. A derivation of the equation for charge transport in a magnetized plasma with particle species momentum transports as the starting equations (see any basic plasma text-book) proves (8) to be all too simplified. A point seldom made in textbooks, however, is that "freezing" of magnetic flux in the limit $\sigma \to \infty$ must always refer primarily to the plasma electron gas frame, not the mass (ionic) frame, see [6].

4. The Hall Term

Equation (8) contains the mass velocity $\boldsymbol{V}$ instead of the plasma electron gas velocity. Textbooks derive the first order correction for such a confusion of frames in the form of the addition to (8) of the Hall term

$$\boldsymbol{j} = \sigma(\boldsymbol{E} + \boldsymbol{V} \times \boldsymbol{B}) - \mu_e B(\boldsymbol{j} \times \boldsymbol{B}/B)\,. \tag{9}$$

μ_e is the electronic mobility and $\mu_e B$ the so-called Hall parameter. It is usually expressed as the product of the electron magnetic Larmor angular frequency ω_{ge} and its collision time τ_e:

$$\mu_e B = \omega_{ge}\tau_e\,, \quad \mu_e = e\tau_e/m_e\,, \quad \omega_{ge} = eB/m_e\,. \tag{10}$$

With the electrons free to perform magnetic Larmor gyrations, as they do in any not too dense and magnetized plasma, the Hall current term in (9) can exceed the left hand side parallel current term by an order of magnitude or more. This is a generally observed, almost trivial, fact in treaties on MHD generators and MPD accelerators. Published fusion plasma theory, however, usually concerns instabilities, anomalities or singularities with respect to ideal or resistive MHD theory.

The condition for neglecting under non-static conditions the Hall term brings back the ion collisionless skin depth λ_i and the plasma characteristic scale length L, cf. (2). The criterion for neglecting the Hall term is [2]

$$\lambda_i/L \ll 1\,, \quad \lambda_i^2 = m_i/\mu_0 e^2 n_i\,. \tag{11}$$

Two observations should be made here.

i) The criterion may be well satisfied in the dense interior region of a magnetically confined high-density plasma. However, the decrease of plasma density towards the plasma boundaries causes the local λ_i-value sooner or later to exceed any characteristic length L of the plasma entity, and then to go to infinity when reaching the vacuum or neutral gas region surrounding the plasma. In other word, ideal or resistive MHD theory breaks down in the boundary region of any high-density plasma entity.

ii) The pertinent dimension and densities of the plasmas in present magnetic fusion research assemblies of the "main-line" types do not satisfy the criterion (11) for resistive or ideal MHD theory validity. For reasonable stability they have to operate at too low density with, approximately, $L \simeq \lambda_i$ as their limit for the obtainable plasma density in a quasi steady-state operation. Fully ionized deuterium of density $10^{13}\,\mathrm{cm}^{-3}$ yields $\lambda_i = 10\,\mathrm{cm}$. Generally, increased plasma volume size requires decreased plasma density, with the line density of straight and toroidally bent discharges being of the order the inverse classical ion, not electron, radius.

5. Elliptic MHD Theory

Equation (1) proves that static plasma confinement requires the current density to be directed at an angle to the magnetic field. This angle may be small if the magnetic pressure is much larger than the plasma pressure, i.e. there is a β-value $\ll 1$. Then, as a limit,

$$\mu_0 \boldsymbol{j} = \lambda \boldsymbol{B} . \tag{12}$$

If, further, the factor λ is taken to be a constant, (5), (6) and (12) yield the elliptic equation

$$\nabla^2 \boldsymbol{B} + \lambda^2 \boldsymbol{B} = 0 . \tag{13}$$

Usually (13) is referred as to follow from the spontaneous relaxation of a plasma magnetic field configuration into a minimum energy state [7] subject to the invariance of a certain volume integral, denoted the helicity K_0, and taken over the configuration as

$$K_0 = \int \boldsymbol{A} \cdot \boldsymbol{B}\, \mathrm{d}v, \quad \boldsymbol{B} = \operatorname{curl} \boldsymbol{A} . \tag{14}$$

A combination of (12) and (14) directly proves that the "minimized" magnetic energy is proportional to the invariant

$$W_m = \tfrac{1}{2} \int \boldsymbol{j} \cdot \boldsymbol{A}\, \mathrm{d}v = \lambda K_0 / 2\mu_0 . \tag{15}$$

In [8] modificiations of K_0 are discussed in order to make it physically meaningful, consistent and gauge invariant. Noting the postulate character in taking λ equal to a constant, [1] rejects elliptic MHD theory outright.

6. The Physics Behind the HMHD Theory

As implied by the acronym HMDH, also standing for Hall effect Magneto Hydro Dynamics, it is a magnetized plasma description that properly accounts for the Hall effect, in contrast to the ideal or resistive MHD theory. However, just the addition of the Hall term in (9) is no more than adding an important, missing term.

The MHD theories discussed so far involve *three* Maxwell field equations, (4), (5) and (6). HMHD theory derives from the inclusion also of the *fourth* Maxwell equation

$$\varepsilon_0 \operatorname{div} \boldsymbol{E} = \varrho_e . \tag{16}$$

ε_0 is the permittivity of free space and ϱ_e is the charge density, i.e. the plasma excess charge.

The characteristic property of the plasma state is its absence of macroscopic charge accumulation, i.e.

$$\varrho_e = e(n_i - n_e) \simeq 0 \tag{17}$$

for singly ionized ions. Basic treaties on plasmas simply dismiss (16) as redundant because of (17). [9] is an outstanding exception to this, stressing the fundamental nature of the seemingly contradictory "plasma approximation" $n_i = n_e$ but $\operatorname{div} \boldsymbol{E} \neq 0$ at the same time.

The HMHD plasma description follows by imposing the charge neutrality restriction upon the electromagnetic torque distribution between the two plasma particle species, electrons and ions. For the details of this derivation see [2]. Essentially, it is just the Penfield-Haus [10] stress tensors symmetry analysis applied to a magnetized plasma subject to the charge neutrality restriction. The central out-come is the preferential torque and thereby angular momentum transfer by the electromagnetic field upon the *ionic* species in case the plasma exhibits Hall effect, i.e. when the electrons are free to perform Larmor gyrations.

HMHD is inherently a two-fluids plasma description. It provides the theoretical justification for the well-known and succesful "ad hoc" Ion Vlasov Fluid Model [11], where the ions are taken to respond dynamically to fields and forces but the electrons are seen just as a mass-less and charge-neutralizing background. Such a plasma behaviour conflicts with the long-standing belief that quasi-neutrality implies an equipartition as to magnitude between electrons and ions in the mechanical angular momentum transfer to them from a rotational electric field. That belief, actually a claim for the validity of Newton's third law, is totally incorrect in contexts of charged particle electrodynamics [12]. Angular momentum also and always resides in the electromagnetic field, i.e. the relevant variable is the canonical, not mechanical, angular momentum, and charge neutrality can be sustained only by non-central particle-particle interaction forces, in violation of Newton's third law.

HMHD thus explains an Ion Vlasov Fluid characteristic of the decoupling in the rotational motion of the plasma species. From the simple electron motion equation

$$e n_e (\boldsymbol{E} + \boldsymbol{V}_e \times \boldsymbol{B}) = 0 \tag{18}$$

a curl operation and (4) yield

$$\partial \boldsymbol{B}/\partial t - \operatorname{curl}(\boldsymbol{V}_e \times \boldsymbol{B}) = 0\,, \tag{19}$$

which expresses the Lighthill [6] theorem of approximate magnetic flux conservation in the electron gas frame, not mass frame. However, the ion transport counterpart to (18) must include inertia,

$$n_i m_i \, \mathrm{d}\boldsymbol{V}_i/dt = e n_i (\boldsymbol{E} + \boldsymbol{V}_i \times \boldsymbol{B})\,, \tag{20}$$

This equation implies

$$\begin{aligned} \partial \boldsymbol{\Omega}_i/\partial t &= \operatorname{curl}(\boldsymbol{V}_i \times \boldsymbol{\Omega}_i)\,, \\ \boldsymbol{\Omega}_i &= \operatorname{curl} \boldsymbol{V}_i + \mathrm{e}\boldsymbol{B}/m_i\,, \end{aligned} \tag{21}$$

which expresses conservation of *canonical,* i.e. matter-plus-field angular momentum. This property is the central feature of the HMHD plasma description. Physically, it just describes how the torque associated with the inductive electric field in the moving ionic mass frame acts preferentially on the ion gas so to speed up or retard rotation.

An important theoretical result has recently been obtained about plasmas obeying HMHD theory, see [13]. By use of variational calculus in minimizing the total field-plus-kinetic energy of an incompressible cylindrical plasma a general relaxation towards a non-force free final stage was proved with the ion plasma wave-length $\lambda_i = c/\omega_{pi}$ as the natural scale length, cf. (2) and (11).

Indeed, observations seem to support this important results, in conflict with ideal or resistive MHD theory. Magnetized plasmas in astrophysics exhibit rotation rather as a rule than as an exception, and the remarkable resilience of laboratory rotating ion plasma rings has been well known but not well understood.

7. Hyperbolic Magnetohydrodynamics, HMHD

Equations (19) expresses the approximate conservation of the canonical vorticity $\boldsymbol{\Omega}_i$ along the streamlines of motion defined by the velocity $\boldsymbol{V}_i$. If, initially, $\boldsymbol{\Omega}_i = 0$ it remains so, and

$$\operatorname{curl} \boldsymbol{V}_i = e\boldsymbol{B}/m_i\,. \tag{22}$$

With electrons "frozen" to field lines the current density is taken to be ionic:

$$\operatorname{curl} \boldsymbol{B} = \mu_0 e n_i \boldsymbol{V}_i\,. \tag{23}$$

This equation pair (22) and (23), resembling the London equations of superconductivity theory, directly yields a plasma Meissner effect differential equation of hyperbolic type,

$$\nabla^2 \boldsymbol{B} - \lambda_i^{-2} \boldsymbol{B} = 0 \tag{24}$$

for moderate variations in plasma density, grad $\ln n_i \simeq 0$. The quantum supercurrents of the Meissner effect correspond in the plasma case to inductively driven ion currents, forming separate magnetized whirl structures, eventually with equipartition between field and kinetic energies, in accordance with the theoretical relaxation prediction of [13].

As mentioned, any "magnetic bottle" configuration for confining a high density plasma entity has to include a plasma edge region of lower density surrounding the higher density inner parts. This implies a crucial enclosure region in which the central region diffusive MHD theory, c.f. (7), crosses over into its hyperbolic counterpart, c.f. (24). Indeed, a bewildering multitude of plasma edge or boundary phenomena (H- and L-mode transitions, particle emissions, magnetic island formation, radiation bands, etc.) appears to be characteristic property and main obstacle in magnetic confinement research [14].

From the theory of [13] and the analogy to superconductivity phenomena one may expect, in particular, the spontaneous formation in the edge region of filaments, actually diamagnetic whirl structures, superimposed upon the initial confining field. Being "fed" by this field such whirls "eat" their way into regions, of higher plasma density and higher field strength, ultimately ruining any plasma confinement. So-called m-number instabilities of just this character normally occur in experiments to confine high-density plasmas magnetically [15].

The hyperbolic HMHD theory concerns the shifting between the field and the matter parts of the approximately conserved canonical angular momentum for the plasma heavy species. Of course, this shifting has its counterpart as to energies involved, in particular observed transfers of magnetic field energy into kinetic energies such as mass motions, particle emissions, heating and turbulence. Such phenomena are usually referred to by the expression "magnetic field line reconnection" and "explained" in terms of diffusive or elliptic MHD theory augmented by anomalies, instabilities or, preferably, singularities, see e.g. four extensive reviews in [16].

8. In Conclusion, Non-thermal Magnetic Fusion

The properties and performance of the dense plasma focus differ drastically compared to the magnetic confinement thermal fusion [17]. Under optimized conditions its fusion neutrons emission exceeds that of a corresponding thermalized plasma by one or more orders of magnitude, already allowing for realistic extrapolations, based on experiments, up to fusion reactor levels [18]. Most neutrons are due to the interaction between relatively low density ($10^{17}-10^{18}\,\mathrm{cm}^{-3}$) plasma structures and medium energy (less than 100 keV) ions confined for a long period of time in a self-sustained magnetic configuration. The accelerating electric field can only be of inductive origin, and they arise upon formation or disruption of intensely magnetized HMHD whirl structures ("Bostick whirls") theoretically obtainable by a generalization of the Bennett relation [19].

According to the generally accepted thinking, fundamental for magnetic confinement fusion, beam-target nuclear energy production is ruled out by higher energy cost of beam than energy yield from fusion. This result has been taken as an almost trivial consequence of the very small probability or cross section for a nuclear fusion reaction compared to those for scatterings. The copious fusion neutrons emission from plasma foci, caused by reactions with magnetically confined ions beams, prove a contradiction to this very foundation about the necessity for *thermo*-nuclear fusion energy. The HMHD plasma description provides the explanation: The simple scattering-thermalization arguments do not apply to ion beams in self-sustained magnetic fields and obeying the conservation of their *canonical* angular momentum. For such ions the energy of their field part momentum balances gains or losses in their matter part [14]. Non-thermal magnetic fusion becomes one of the Emerging Nuclear Energy Systems.

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